

# AdS-inspired noncommutative gravity on the Moyal plane

Marija Dimitrijević<sup>1</sup>, Voja Radovanović<sup>1</sup> and Hrvoje Štefančić<sup>2</sup>

<sup>1</sup>University of Belgrade, Faculty of Physics  
Studentski trg 12, 11000 Beograd, Serbia

<sup>2</sup>Rudjer Bošković Institute, Theoretical Physics Division  
Bijenička 54, 10002 Zagreb, Croatia

## Abstract

We consider noncommutative gravity on a space with canonical noncommutativity that is based on the commutative AdS gravity. Gravity is treated as gauge theory of the noncommutative  $SO(1,3)_*$  group and the Seiberg-Witten (SW) map is used to express noncommutative fields in terms of the corresponding commutative fields. In the commutative limit the noncommutative action reduces to the Einstein-Hilbert action plus the cosmological term and the topological Gauss-Bonnet term. After the SW expansion in the noncommutative parameter the first order correction to the action, as expected, vanishes. We calculate the second order correction and write it in a manifestly gauge covariant way.

**Keywords:** gauge theory of gravity, canonical noncommutativity, Seiberg-Witten map

eMail: dmarija,rvoja@ipb.ac.rs; shrvoje@thphys.irb.hr

# 1 Introduction

Field theories on noncommutative (NC) spaces have been investigated in many aspects during the last twenty years. Various approaches to definition and analysis of the properties of noncommutative spaces are present in the literature [1]. One of the most frequently used is the approach of deformation quantization [2]. In this approach noncommutative functions  $\hat{f}(\hat{x})$  are mapped to the functions of commuting coordinates  $f(x)$  and the abstract algebra multiplication is represented by  $\star$ -product, which is a deformation of the usual point-wise multiplication. The simplest and the most analyzed example of the  $\star$ -product is the Moyal-Weyl  $\star$ -product

$$f(x) \star g(x) = e^{\frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \frac{\partial}{\partial y^\nu}} f(x) g(y)|_{y \rightarrow x} , \quad (1.1)$$

defined by a constant antisymmetric matrix  $\theta^{\mu\nu}$ . Using this type of deformation various problems were investigated: NC scalar field theories, NC gauge theories, deformations of supersymmetric theories,...

An important boost to the formulation of NC gauge theories came with the paper of Seiberg and Witten [3]. They found a connection between gauge theories on the commutative space and the corresponding NC gauge theories. This result was then used by Wess et al. [4] to formulate the enveloping algebra approach to NC gauge theories. Namely, for groups which are of importance for physical applications like  $SU(N)$ , NC gauge transformations close in the enveloping algebra. This implies that the NC gauge field is also enveloping algebra-valued which leads to (infinitely many) new degrees of freedom. However, using the Seiberg-Witten (SW) map one can express all enveloping algebra-valued NC variables (gauge parameter and fields) in terms of the corresponding commutative variables. In that way both theories have the same number of degrees of freedom. This approach enabled the analysis of renormalizability of NC gauge theories [5], anomalies [6], construction of a NC deformation of the Standard Model [7] and investigation of its phenomenological consequences [8].

On the other hand, construction of a NC generalization of General Relativity (GR) proved to be a difficult task. One of the reasons for this is the underlying symmetry of GR, the diffeomorphism symmetry. One can follow the twist approach in which the commutative diffeomorphisms are replaced by the twisted diffeomorphisms [9, 10]. However, a full understanding of the twisted symmetries is still missing. Having in mind that the SW approach works very well for NC gauge theories, many authors consider NC gravity as a gauge theory of the Lorentz/Poincaré group. This can be done in the Einstein-Cartan formalism [11]. In this formalism the gauge field for the local Lorentz symmetry, the spin connection  $\omega = \omega_\mu dx^\mu$  and the vielbein  $e = e_\mu dx^\mu$  are independent fields. The action is given by

$$S_{EC} = \frac{1}{16\pi G_N} \int d^4x \epsilon_{abcd} \epsilon^{\mu\nu\rho\sigma} R_{\mu\nu}^{ab} e_\rho^c e_\sigma^d, \quad (1.2)$$

with the curvature tensor  $R_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu - i[\omega_\mu, \omega_\nu]$ . Indices  $a, b, \dots$  are local Lorentz (or flat) indices, while indices  $\mu, \nu, \dots$  are Einstein or space-time indices. The

condition that the torsion vanishes,  $T_{\mu\nu}^a = 0$  follows from the equations of motion. Using this condition one can express the spin connection in terms of the vielbeins and the theory reduces to the General Relativity with the metric  $g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b$ . The local Lorentz symmetry can be generalized to the NC local Lorentz symmetry in different ways. In [12] a deformation of the Einstein-Hilbert action is proposed. The construction is based on the Wigner-Inonü contraction of the noncommutative  $SO(1,4)_*$  gauge symmetry. However, noncommutative correction to the gravity action was not found. In [13] the starting point was  $GL(2, C)_*$  NC gauge symmetry. This theory in the commutative limit gives a gravity theory with complex metric. In [14] additional conditions (like vanishing torsion in the lowest order of the NC parameter) are used to break the  $U(2,2)_*$  symmetry to the  $SO(1,3)_*$ . Some authors formulate a deformation of gravity theory on NC spaces with space-time dependent noncommutativity [15, 16]. Coupling of the NC gravity described by (1.2) with fermions and gauge fields was discussed in [17] and [18]. It was shown there that if reality of the NC gravity action is imposed, the first order correction (in the NC parameter) has to vanish. The first non-vanishing correction is the second order correction. So far, no complete result for the second order correction was found, mainly because the calculations are too involved.

In this paper we follow the approach of [19]. Our starting point is the local  $SO(2,3)$  symmetry on four dimensional Minkowski space-time. The group  $SO(2,3)$  has 10 generators, leading to the 10 gauge fields in the theory. After the symmetry breaking, the symmetry reduces to the  $SO(1,3)$  and the generators split into 6 fields which correspond to the spin connection and 4 vielbeins. The advantage of this approach is that the vielbeins are not introduced by hand, but they naturally appear from the initial  $SO(2,3)$  symmetry after the symmetry breaking. This set-up is generalized to the case of the Moyal ( $\theta$ -constant) NC space. In the next section we describe in details the commutative  $SO(2,3)$  gravity theory. In Section 3 the NC  $SO(2,3)_*$  gauge theory via the SW map is introduced. As we will see, a problem arises when symmetry breaking is imposed and we discuss possible solutions. The NC gravity action is presented in Section 4. It consists of three terms: the Gauss-Bonnet term, the Einstein-Hilbert term and the cosmological constant term. We calculate the second order corrections for these actions. Especially, we write the second order corrections in a manifestly gauge covariant way. In the last section we analyze our results and discuss various applications, remaining problems and the future work.

## 2 AdS gauge theory on commutative spacetime

The group  $SO(2,3)$  is the isometry group of anti-de Sitter space. Anti-de Sitter space is a maximally symmetric space with negative constant curvature. The  $so(2,3)$  algebra or the AdS algebra consists of 10 generators,  $M_{AB}$ . Indices  $A, B, \dots$  take values 0, 1, 2, 3, 5. The generators obey the following commutation relations:

$$[M_{AB}, M_{CD}] = i(\eta_{AD}M_{BC} + \eta_{BC}M_{AD} - \eta_{AC}M_{BD} - \eta_{BD}M_{AC}) . \quad (2.3)$$

The  $5D$  metric is  $\eta_{AB} = \text{diag}(+, -, -, -, +)$ . A representation of this algebra can be constructed starting from the Clifford generators  $\Gamma_A$  in  $5D$  Minkowski space, which satisfy

$$\{\Gamma_A, \Gamma_B\} = 2\eta_{AB} . \quad (2.4)$$

Then the generators  $M_{AB}$  are

$$M_{AB} = \frac{i}{4}[\Gamma_A, \Gamma_B] . \quad (2.5)$$

If by  $\gamma_a$ ,  $a = 0, 1, 2, 3$ , we denote the gamma matrices in four dimensional Minkowski space  $M_4$  then the gamma matrices in  $5D$  are  $\Gamma_A = (i\gamma_a\gamma_5, \gamma_5)$ .  $\gamma_5$  is defined by  $\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ . It is easy to show that

$$\begin{aligned} M_{ab} &= \frac{i}{4}[\gamma_a, \gamma_b] = \frac{1}{2}\sigma_{ab} , \\ M_{5a} &= \frac{1}{2}\gamma_a . \end{aligned} \quad (2.6)$$

If we introduce momenta  $P_a = \frac{1}{l}M_{a5}$ , where  $l$  is a constant with the dimension of length the AdS algebra (2.3) becomes

$$\begin{aligned} [M_{ab}, M_{cd}] &= i(\eta_{ad}M_{bc} + \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac}) \\ [M_{ab}, P_c] &= i(\eta_{bc}P_a - \eta_{ac}P_b) \\ [P_a, P_b] &= -i\frac{1}{l^2}M_{ab} . \end{aligned} \quad (2.7)$$

In the limit  $l \rightarrow \infty$  the AdS algebra reduces to the usual Poincare algebra in  $M_4$ . This is the Wigner-Inonü contraction of the AdS algebra. Useful relations concerning the algebra and the traces of  $\gamma$  matrices are given in Appendix A.

Let us assume that the space-time has the structure of the 4 dimensional Minkowski space  $M_4$  and follow the usual steps for constructing a gauge theory on  $M_4$ . To each point of  $M_4$  we attach a tangent space representing a copy of anti-de Sitter space AdS. The AdS group  $SO(2, 3)$  acts on matter fields in the tangent space as a group of internal symmetries. The gauge field takes values in the AdS algebra,

$$\omega_\mu = \frac{1}{2}\omega_\mu^{AB}M_{AB} = \frac{1}{4}\omega_\mu^{ab}\sigma_{ab} - \frac{1}{2}\omega_\mu^{a5}\gamma_a . \quad (2.8)$$

The gauge potential  $\omega_\mu^{AB}$  decomposes into  $\omega_\mu^{ab}$  and  $\omega_\mu^{a5}$ . The transformation law of the  $SO(2, 3)$  potential is given by

$$\delta_\epsilon \omega_\mu = \partial_\mu \epsilon - i[\omega_\mu, \epsilon] , \quad (2.9)$$

or in components

$$\begin{aligned} \delta_\epsilon \omega_\mu^{AB} &= \partial_\mu \epsilon^{AB} - \epsilon^A_C \omega_\mu^{CB} + \epsilon^B_C \omega_\mu^{CA} , \\ \delta_\epsilon \omega_\mu^{ab} &= \partial_\mu \epsilon^{ab} - \epsilon^a_c \omega_\mu^{cb} + \epsilon^b_c \omega_\mu^{ca} - \epsilon^a_5 \omega_\mu^{5b} + \epsilon^b_5 \omega_\mu^{5a} , \\ \delta_\epsilon \omega_\mu^{a5} &= \partial_\mu \epsilon^{a5} - \epsilon^a_c \omega_\mu^{c5} + \epsilon^5_c \omega_\mu^{ca} . \end{aligned} \quad (2.10)$$

The field strength is defined in the usual way by

$$F_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu - i[\omega_\mu, \omega_\nu] = \frac{1}{2} F^{AB} M_{AB} . \quad (2.11)$$

Just like the gauge potential, the components of the field strength tensor,  $F_{\mu\nu}^{AB}$  can be split into  $F_{\mu\nu}^{ab}$  and  $F_{\mu\nu}^{a5}$ . It is easy to show that

$$F_{\mu\nu} = \left( R_{\mu\nu}^{ab} - \frac{1}{l^2} (e_\mu^a e_\nu^b - e_\mu^b e_\nu^a) \right) \frac{\sigma_{ab}}{4} - F_{\mu\nu}^{a5} \frac{\gamma_a}{2} , \quad (2.12)$$

where

$$R_{\mu\nu}^{ab} = \partial_\mu \omega_\nu^{ab} - \partial_\nu \omega_\mu^{ab} + \omega_\mu^{ac} \omega_\nu^{cb} - \omega_\mu^{bc} \omega_\nu^{ca} \quad (2.13)$$

$$l F_{\mu\nu}^{a5} = D_\mu e_\nu^a - D_\nu e_\mu^a = T_{\mu\nu}^a . \quad (2.14)$$

Under the local AdS transformation the field strength transforms as

$$\delta_\epsilon F_{\mu\nu} = i[\epsilon, F_{\mu\nu}] , \quad (2.15)$$

or more explicitly

$$\begin{aligned} \delta_\epsilon F_{\mu\nu}^{ab} &= -\epsilon^{ac} F_{\mu\nu c}^b + \epsilon^{bc} F_{\mu\nu c}^a - \epsilon^{a5} F_{\mu\nu 5}^b + \epsilon^{b5} F_{\mu\nu 5}^a \\ \delta_\epsilon T_{\mu\nu}^a &= -\epsilon^{ac} T_{\mu\nu c}^a + \epsilon^{5c} F_{\mu\nu c}^a . \end{aligned} \quad (2.16)$$

Looking at equations (2.8), (2.10), (2.12) and (2.16) one is tempted to put  $\epsilon^{a5} = 0$  and identify  $\omega_\mu^{ab}$  with the spin connection of the Poincaré gauge theory,  $\omega_\mu^{a5}$  with the vielbeins,  $R_{\mu\nu}^{ab}$  with the curvature tensor and  $F_{\mu\nu}^{a5}$  with the torsion.

Indeed, it was shown in the seventies that one can really do such an identification and relate AdS gauge theory with GR. One way was introduced by MacDowel and Mansouri [20]. They start from the  $SO(2,3)$  gauge theory but make an additional assumption: that all fields in the theory transform covariantly under the action of infinitesimal diffeomorphisms. The action is written in a way which breaks the  $SO(2,3)$  gauge symmetry down to  $SO(1,3)$  and it is invariant under the infinitesimal diffeomorphisms. Then one can identify  $\omega_\mu^{a5}$  with the vielbein and after going to the second order formalism obtain GR<sup>1</sup>. A similar approach was discussed by Townsend in [21].

A more elegant way of relating AdS gauge theory with GR was introduced by Stelle and West [19]. They also start from the  $SO(2,3)$  gauge theory, but they spontaneously break it down to  $SO(1,3)$ . Their starting action is invariant under the full  $SO(2,3)$  gauge symmetry and they introduce one additional auxiliary field in order to perform the symmetry breaking. In a particular gauge, their action reduces to the MacDowel-Mansouri action which is invariant under the  $SO(1,3)$  gauge symmetry and again the fields  $\omega_\mu^{a5}$  can be interpreted as vielbeines. In other gauges the  $SO(2,3)$  symmetry is realized nonlinearly, while the  $SO(1,3)$  subgroup is realized linearly. In that way the

---

<sup>1</sup>This holds if there are no spinors in the theory. If the spinor fields appear, the torsion is nonzero and the pure gravity part of the theory does not reduce to GR.

diffeomorphism invariance follows from the spontaneous symmetry breaking (SSB) and does not have to be introduced by hand at the very beginning.

Now we focus on constructing the  $SO(2, 3)$  gauge invariant action following the Stelle-West approach. The action invariant under the  $SO(2, 3)$  gauge transformations is given by

$$S = \frac{i l}{64\pi G_N} \text{Tr} \int d^4x \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \phi + \lambda \int d^4x \left( \frac{1}{4} \text{Tr} \phi^2 - l^2 \right), \quad (2.17)$$

where  $G_N$  is the Newton gravitational constant and  $\lambda$  is the Lagrange multiplier and an additional auxiliary field  $\phi = \phi^A \Gamma_A$  transforming in the adjoint representation of  $SO(2, 3)$  is introduced

$$\delta\phi = i[\epsilon, \phi]. \quad (2.18)$$

The field  $\phi$  is constrained by the condition  $\phi_A \phi^A = l^2$ . Choosing  $\phi^a = 0$ ,  $\phi^5 = l$  the  $SO(2, 3)$  symmetry is broken spontaneously to  $SO(1, 3)$  and we obtain the action

$$\begin{aligned} S &= \frac{i l^2}{64\pi G_N} \epsilon^{\mu\nu\rho\sigma} \int d^4x \text{Tr}(F_{\mu\nu} F_{\rho\sigma} \gamma_5) \\ &= -\frac{1}{16\pi G_N} \int d^4x \left[ \frac{l^2}{16} \epsilon^{\mu\nu\rho\sigma} \epsilon_{abcd} R_{\mu\nu}^{ab} R_{\rho\sigma}^{cd} + eR + 2e\Lambda \right], \end{aligned} \quad (2.19)$$

where  $\Lambda = -3/l^2$  and  $e = \det(e_\mu^a)$ . In the first line we inserted expansions (2.12) and (2.13) and after some standard manipulation with indices and traces we obtained the second line.

The vielbeins and spin connection are independent variables and in addition, the spin connection does not propagate. The last statement follows from the action (2.19). Varying the action with respect to the spin connection we obtain an equation which relates connection and vielbein. In this way we can express the spin connection in terms of the vielbein. Since there is no fermionic matter in the action (2.19) this equation gives vanishing of the torsion. In that case the first term in (2.19) is the Gauss-Bonnet term; it is a topological term and does not contribute to the equations of motion. The second term is the Einstein-Hilbert action, while the last term is the cosmological constant term. From the vielbeins  $e_\mu^a$  we can construct the metric tensor

$$g_{\mu\nu} = \eta_{ab} e_\mu^a e_\nu^b. \quad (2.20)$$

The action (2.19) is invariant under the  $SO(1, 3)$  gauge transformations. This action is in addition invariant under the infinitesimal diffeomorphisms as a consequence of spontaneous symmetry breaking, as discussed in [19].

### 3 NC $SO(2, 3)_*$ gauge theory

In this section we try to generalize the model (2.17) to the NC case. We will not discuss the Seiberg-Witten map and related calculations in details. Instead we will show that

the mechanism of spontaneous symmetry breaking (SSB) does not work in this case and we will suggest a solution to this problem .

We work with the simplest form of noncommutativity, canonical or  $\theta$ -constant noncommutativity,

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad (3.21)$$

with the constant antisymmetric matrix  $\theta^{\mu\nu}$ . Following the approach of deformation quantization we represent noncommutative functions as functions of commuting coordinates and algebra multiplication with the Moyal-Weyl  $\star$ -product (1.1)

$$\begin{aligned} \hat{f}(\hat{x}) &\mapsto f(x) \\ \hat{f}(\hat{x})\hat{g}(\hat{x}) &\mapsto (f \star g)(x). \end{aligned}$$

In order to construct the NC  $SO(2, 3)_\star$  gauge theory we use the enveloping algebra approach and the Seiberg-Witten map developed in [4]. Under the infinitesimal NC  $SO(2, 3)_\star$  gauge transformations the NC gauge field  $\hat{\omega}_\mu = \frac{1}{2}\hat{\omega}_\mu^{AB}M_{AB}$  transforms as

$$\delta_\epsilon^\star \hat{\omega}_\mu = \partial_\mu \hat{\Lambda}_\epsilon + i[\hat{\Lambda}_\epsilon \star \hat{\omega}_\mu], \quad (3.22)$$

with the NC gauge parameter  $\hat{\Lambda}_\epsilon$ . These transformations close the algebra

$$[\delta_{\epsilon_1}^\star \star \delta_{\epsilon_2}^\star] = \delta_{-i[\epsilon_1, \epsilon_2]}^\star, \quad (3.23)$$

provided that the gauge parameter  $\hat{\Lambda}_\epsilon$  is in the enveloping algebra of the  $so(2, 3)$  algebra. The basic idea of the Seiberg-Witten map is that all noncommutative variables (gauge parameter, fields) can be expressed in terms of the corresponding commutative variables as power series in noncommutativity (NC) parameter  $\theta^{\mu\nu}$ . In case of the NC gauge parameter the expansion is

$$\hat{\Lambda}_\epsilon = \Lambda^{(0)} + \Lambda^{(1)} + \dots \quad (3.24)$$

Zeroth order solution is just the commutative gauge parameter  $\epsilon = \frac{1}{2}\epsilon^{AB}M_{AB}$ . First order solution (and all higher order solutions) follows from equation (3.23) and it is given by

$$\hat{\Lambda}^{(1)} = -\frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, \partial_\beta \epsilon\}, \quad (3.25)$$

where  $\omega_\mu = \frac{1}{2}\omega_\mu^{AB}M_{AB}$  is the commutative gauge potential. Using the solutions of the SW map given in [22] one can write the gauge field  $\hat{\omega}_\mu$ , the field strength tensor  $\hat{F}_{\mu\nu}$  and the field  $\hat{\phi}$  transforming in the adjoint representation in terms of the corresponding commutative fields  $\omega_\mu$ ,  $F_{\mu\nu}$  and  $\phi$ . Then the action (2.17) is generalized to

$$S = \frac{i l}{64\pi G_N} \text{Tr} \int d^4x \epsilon^{\mu\nu\rho\sigma} \hat{F}_{\mu\nu} \star \hat{F}_{\rho\sigma} \star \hat{\phi} + \lambda \int d^4x \left( \frac{1}{4} \text{Tr} \hat{\phi} \star \hat{\phi} - l^2 \right). \quad (3.26)$$

The field  $\hat{\phi}$  is constrained by<sup>2</sup>  $\hat{\phi}\hat{\phi} = l^2$ . Using this constraint to break symmetry from  $SO(2,3)_\star$  to  $SO(1,3)_\star$  implies the action

$$S = \frac{il^2}{64\pi G_N} \text{Tr} \int d^4x \epsilon^{\mu\nu\rho\sigma} \hat{F}_{\mu\nu} \star \hat{F}_{\rho\sigma} \gamma_5 . \quad (3.28)$$

This action is supposed to have the NC  $SO(1,3)_\star$  gauge symmetry. Unfortunately, it has not. Therefore, when expanded in orders of NC parameter using the SW map it will also not have the commutative  $SO(1,3)$  symmetry and therefore it will not be possible to reconstruct GR in the commutative limit. The easiest way to see why the mechanism of SSB fails in this case is to look at the gauge parameter  $\hat{\Lambda}_\epsilon$ . If we put  $\epsilon^{a5} = 0$  in the solution of the SW map for  $\hat{\Lambda}_\epsilon$  (3.25) we obtain

$$\begin{aligned} \hat{\Lambda}_{so(2,3)}^{(1)} &= -\frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, \partial_\beta\epsilon\} \\ &= -\frac{1}{4}\theta^{\alpha\beta}\left\{\frac{1}{4}\omega_\alpha^{ab}\sigma_{ab} - \frac{1}{2}\omega_\alpha^{a5}\gamma_a, \frac{1}{4}\partial_\beta\epsilon^{cd}\sigma_{cd}\right\} \\ &= -\frac{1}{64}\theta^{\alpha\beta}\omega_\alpha^{ab}\partial_\beta\epsilon^{cd}\{\sigma_{ab}, \sigma_{cd}\} + \frac{1}{32}\theta^{\alpha\beta}\omega_\alpha^{a5}\partial_\beta\epsilon^{cd}\{\gamma_a, \sigma_{cd}\} . \end{aligned}$$

The second term in the last line produces a term proportional to  $\gamma_a\gamma_5$  which is not in the enveloping algebra of  $SO(1,3)$ . Remember that in this representation the enveloping algebra of  $SO(1,3)$  consists of  $\sigma_{ab}$ ,  $I$  and  $\gamma_5$ . More explicitly, the SW map for the  $SO(1,3)_\star$  gauge symmetry gives

$$\hat{\Lambda}_{so(1,3)}^{(1)} = -\frac{1}{64}\theta^{\alpha\beta}\omega_\alpha^{ab}\partial_\beta\epsilon^{cd}\{\sigma_{ab}, \sigma_{cd}\} \quad (3.29)$$

and there is no  $\gamma_a\gamma_5$ -valued term.

A way to solve this problem is to expand the action (3.26) up to second order in the NC parameter using the SW map solutions first. The expanded action has commutative  $SO(2,3)$  symmetry which can then be broken to commutative  $SO(1,3)$  symmetry. The obtained action could then be analyzed. This work we postpone for the next publication.

## 4 AdS inspired NC gravity

We saw that one cannot spontaneously break the NC  $SO(2,3)_\star$  gauge symmetry and obtain GR in the commutative limit. However, one can still choose the action (2.19) as the starting point and work with the NC  $SO(1,3)_\star$  gauge symmetry from the very beginning. Then the SW map guarantees that the expanded action will have commutative  $SO(1,3)$  gauge symmetry. That enables the reconstruction of GR in the commutative limit.

---

<sup>2</sup>The integral is cyclic

$$\int d^4x f \star g \star h = \int d^4x h \star f \star g . \quad (3.27)$$

Especially  $\int d^4x f \star g = \int d^4x g \star f = \int d^4x fg$ .



With this motivation, let us write the NC generalization of the action (2.19). It is given by

$$S = \frac{i l^2}{64\pi G_N} \int d^4x \epsilon^{\mu\nu\rho\sigma} \left[ \text{Tr}(\hat{R}_{\mu\nu} \star \hat{R}_{\rho\sigma} \gamma_5) - \frac{i}{l^2} \text{Tr}(\hat{R}_{\mu\nu} \star \hat{E}_\rho \star \hat{E}_\sigma \gamma_5) - \frac{1}{4l^4} \text{Tr}(\hat{E}_\mu \star \hat{E}_\nu \star \hat{E}_\rho \star \hat{E}_\sigma \gamma_5) \right], \quad (4.30)$$

with noncommutative vielbeins  $\hat{E}_\mu$  and noncommutative curvature  $\hat{R}_{\mu\nu}$  defined by

$$\hat{R}_{\mu\nu} = \partial_\mu \hat{\omega}_\nu - \partial_\nu \hat{\omega}_\mu - i[\hat{\omega}_\mu \star \hat{\omega}_\nu], \quad (4.31)$$

where  $\hat{\omega}_\mu$  is the noncommutative  $SO(1,3)_\star$  gauge potential.

#### 4.1 Seiberg-Witten map

Under the deformed gauge transformations the gauge potential and the curvature tensor transform as

$$\begin{aligned} \delta_\epsilon^\star \hat{\omega}_\mu &= \partial_\mu \hat{\Lambda}_\epsilon - i[\hat{\omega}_\mu \star \hat{\Lambda}_\epsilon] \\ \delta_\epsilon^\star \hat{R}_{\mu\nu} &= i[\hat{\Lambda}_\epsilon \star \hat{R}_{\mu\nu}], \end{aligned} \quad (4.32)$$

where  $\hat{\Lambda}_\epsilon$  is the noncommutative gauge parameter. The NC vielbein  $\hat{E}_\mu$  transforms in the adjoint representation

$$\delta_\epsilon^\star \hat{E}_\mu = i[\hat{\Lambda}_\epsilon \star \hat{E}_\mu]. \quad (4.33)$$

Notice that all noncommutative fields belong to the enveloping algebra of  $SO(1,3)$ . For example, the  $\star$ -commutator in (4.33) does not close in the Lie algebra. This means that in the NC theory we have (apparently) infinitely many new degrees of freedom compared with the commutative theory.

We have said before that the problem of additional degrees of freedom is solved by the Seiberg-Witten map. We repeat once again that the basic idea of this map is that all NC variables can be expressed in terms of the corresponding commutative variables and their derivatives. These expressions are power series expansions in the NC parameter

$$\begin{aligned} \hat{\Lambda}_\epsilon &= \epsilon + \hat{\Lambda}^{(1)} + \hat{\Lambda}^{(2)} + \dots, \\ \hat{\omega}_\mu &= \omega_\mu + \hat{\omega}_\mu^{(1)} + \hat{\omega}_\mu^{(2)} + \dots \\ \hat{E}_\mu &= e_\mu + \hat{E}_\mu^{(1)} + \hat{E}_\mu^{(2)} + \dots, \end{aligned}$$

where the higher order corrections are functions of the commutative variables  $\epsilon$ ,  $\omega_\mu$ ,  $e_\mu$  and their derivatives. We will not solve the SW map for our NC fields but use the results already present in the literature [22], [23] and apply them to the case of  $SO(1,3)_\star$  NC gauge group.

The requirement that the commutator of two NC gauge transformations is a NC gauge transformation again

$$[\delta_\alpha^\star \star \delta_\beta^\star] = \delta_{-i[\alpha, \beta]}^\star \quad (4.34)$$

gives the solution for  $\Lambda_\epsilon^{(1)}, \Lambda_\epsilon^{(2)}, \dots$ . The recursive relation between the  $(n+1)$ st order and the  $n$ th order solution is given by

$$\hat{\Lambda}^{(n+1)} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\left(\{\hat{\omega}_\kappa \star \partial_\lambda \hat{\epsilon}\}\right)^{(n)}, \quad (4.35)$$

where  $(A \star B)^{(n)} = A^{(n)}B^{(0)} + A^{(n-1)}B^{(1)} + \dots + A^{(0)}\star^{(1)}B^{(n-1)} + A^{(1)}\star^{(1)}B^{(n-2)} + \dots$  includes all possible terms of order  $n$ . The explicit expressions for  $\hat{\Lambda}_\epsilon^{(1)}$  and  $\hat{\Lambda}_\epsilon^{(2)}$  in the case for the  $SO(1,3)_\star$  gauge group are given in Appendix B. Since  $\epsilon$  and  $\omega_\mu$  contain  $\sigma_{ab}$  matrices then the noncommutative gauge parameter has the following structure

$$\hat{\Lambda}_\epsilon = \frac{1}{4}\Lambda^{ab}\sigma_{ab} + \Lambda I + \Lambda^5\gamma_5. \quad (4.36)$$

This means that  $[\hat{\Lambda}_\epsilon, \gamma_5] = 0$ . Using this fact it is easy to prove that the action (4.30) is invariant under the NC  $SO(2,3)_\star$  gauge group.

Solving the equation

$$\hat{\omega}_\mu(\omega) + \delta_\epsilon^\star \hat{\omega}_\mu(\omega) = \hat{\omega}_\mu(\omega + \delta_\epsilon \omega) \quad (4.37)$$

order by order in the NC parameter we can express noncommutative gauge potential  $\hat{\omega}_\mu$  in terms of the commutative gauge potential  $\omega_\mu$ . The recursive solution in this case is given by

$$\hat{\omega}_\mu^{(n+1)} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\left(\{\hat{\omega}_\kappa \star \partial_\lambda \hat{\omega}_\mu + \hat{R}_{\lambda\mu}\}\right)^{(n)}. \quad (4.38)$$

Looking at the form of this solution, we conclude that the gauge field  $\hat{\omega}_\mu$  has to be of the form

$$\hat{\omega}_\mu = \frac{1}{4}\omega_\mu^{ab}\sigma_{ab} + \omega_\mu I + \omega_\mu^5\gamma_5. \quad (4.39)$$

The solution for the curvature tensor  $\hat{R}_{\mu\nu}$  follows from the definition (4.31). The recursive formula is

$$\begin{aligned} \hat{R}_{\mu\nu}^{(n+1)} &= -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\left(\{\hat{\omega}_\kappa \star \partial_\lambda \hat{R}_{\mu\nu} + D_\lambda \hat{R}_{\mu\nu}\}\right)^{(n)} \\ &\quad + \frac{1}{2(n+1)}\theta^{\kappa\lambda}\left(\{\hat{R}_{\mu\kappa} \star \hat{R}_{\nu\lambda}\}\right)^{(n)} \end{aligned} \quad (4.40)$$

and one can check that

$$\hat{R}_{\mu\nu} = \frac{1}{4}R_{\mu\nu}^{ab}\sigma_{ab} + R_{\mu\nu}I + R_{\mu\nu}^5\gamma_5. \quad (4.41)$$

The noncommutative vielbein transforms in the adjoint representation of the NC  $SO(1,3)_\star$  gauge group

$$\delta_\epsilon^\star \hat{E}_\mu = i[\hat{\Lambda}_\epsilon \star \hat{E}_\mu]. \quad (4.42)$$

The recursive solution is given by

$$\hat{E}_\mu^{(n+1)} = -\frac{1}{4(n+1)}\theta^{\kappa\lambda}\left(\{\hat{\omega}_\kappa \star \partial_\lambda \hat{E}_\mu + D_\lambda \hat{E}_\mu\}\right)^{(n)} \quad (4.43)$$

with  $D_\lambda \hat{E}_\mu = \partial_\lambda \hat{E}_\mu - i[\hat{\omega}_\lambda \star \hat{E}_\mu]$ . The NC vielbein has the structure

$$\hat{E}_\mu = E_\mu^a \gamma_a + E_\mu^5 \gamma_5 . \quad (4.44)$$

## 4.2 Action

Using the Sieberg-Witten map solutions (4.38), (4.40) and (4.43), the action (4.30) becomes a power series expansion in NC parameter  $\theta^{\mu\nu}$

$$\hat{S} = S + S^{(1)} + S^{(2)} + \dots , \quad (4.45)$$

where  $S^{(1)}$  and  $S^{(2)}$  are the first and second order corrections respectively. The zeroth order coincides with commutative action (2.19). The first order correction is given by

$$\begin{aligned} S^{(1)} = & \frac{i l^2}{64\pi G_N} \epsilon^{\mu\nu\rho\sigma} \int d^4x \left( 2\text{Tr}(\hat{R}_{\mu\nu}^{(1)} R_{\rho\sigma} \gamma_5) \right. \\ & - \frac{i}{l^2} \text{Tr}(\hat{R}_{\mu\nu}^{(1)} e_\rho e_\sigma \gamma_5) - \frac{2i}{l^2} \text{Tr}(R_{\mu\nu} \hat{E}_\rho^{(1)} e_\sigma \gamma_5) + \frac{1}{2l^2} \theta^{\alpha\beta} \text{Tr}(R_{\mu\nu} (\partial_\alpha e_\rho) (\partial_\beta e_\sigma) \gamma_5) \\ & \left. - \frac{1}{2l^4} \text{Tr}[e_\mu e_\nu (2e_\rho \hat{E}_\sigma^{(1)} + \frac{i}{2} \theta^{\alpha\beta} \partial_\alpha e_\rho \partial_\beta e_\sigma) \gamma_5] \right) . \end{aligned} \quad (4.46)$$

Inserting the solutions for  $\hat{R}_{\mu\nu}^{(1)}$  and  $\hat{E}_\mu^{(1)}$  and calculating the traces in (4.46) one finds that the first order correction vanishes. In [12] and [17] it was shown that all odd order corrections vanish if the reality of the action is taken into account. Therefore, the first non-vanishing correction is the second order and we have to calculate it explicitly.

The second order correction for the action (4.46) is a sum of three terms

$$S^{(2)} = S_{GB}^{(2)} + S_{EH}^{(2)} + S_\Lambda^{(2)} . \quad (4.47)$$

The first term is a deformation of the Gauss-Bonnet topological term; the second term is a deformation of the Einstein-Hilbert action and the last term is a correction to the cosmological constant term. We will analyze them separately.

### 4.2.1 Gauss-Bonnet action

The second order correction to the Gauss-Bonnet action is given by

$$\begin{aligned} S_{GB}^{(2)} &= \frac{i l^2}{64\pi G_N} \epsilon^{\mu\nu\rho\sigma} \int d^4x \left( \text{Tr}(\hat{R}_{\mu\nu} \star \hat{R}_{\rho\sigma} \gamma_5) \right)^{(2)} \\ &= \frac{i l^2}{64\pi G_N} \epsilon^{\mu\nu\rho\sigma} \int d^4x \left( \frac{i}{2} \epsilon_{abcd} R_{\mu\nu}^{ab} R_{\rho\sigma}^{(2)cd} + 8 R_{\mu\nu}^{(1)} R_{\rho\sigma}^{(1)} \right) , \end{aligned} \quad (4.48)$$

where we used (4.41) and the formulas from Appendix A to calculate the traces. Now one has to insert the explicit solutions for  $\hat{R}_{\mu\nu}^{(1)}$  and  $\hat{R}_{\mu\nu}^{(2)}$  which are given in Appendix B, formulas (2.77) and (2.78) respectively. The calculation is straightforward but very lengthy. It can be considerably simplified using the following trick: We take the

commutative gauge field  $\omega_\mu$  to be a constant field. In the final result we reconstruct the curvature tensor and its covariant derivatives for this specific choice from

$$\begin{aligned}\omega_{\mu c}^a \omega_\nu^{cb} - \omega_{\mu c}^b \omega_\nu^{ca} &\rightarrow R_{\mu\nu}^{ab} \\ \omega_{\mu c}^a R_{\nu\rho}^{cb} - \omega_{\mu c}^b R_{\nu\rho}^{ca} &\rightarrow (D_\mu R_{\nu\rho})^{ac} .\end{aligned}\quad (4.49)$$

Inserting  $\omega_\mu = \text{const}$  into (4.48) leads to

$$\begin{aligned}S_{GB}^{(2)} &= -\frac{l^2}{1024\pi G_N} \theta^{\kappa\lambda} \theta^{\rho\sigma} \epsilon^{\mu\nu\alpha\beta} \epsilon_{abcd} \int d^4x \left[ R_{\alpha\beta}^{cd} R_{\mu\kappa}^{ab} R_{\nu\rho}^{mn} R_{\lambda\sigma mn} \right. \\ &\quad - \frac{1}{2} R_{\alpha\beta}^{cd} R_{\mu\kappa}^{ab} R_{\nu\lambda}^{mn} R_{\rho\sigma mn} + R_{\alpha\beta}^{mn} R_{\mu\kappa mn} R_{\nu\rho}^{ab} R_{\lambda\sigma}^{cd} \\ &\quad \left. + R_{\mu\rho}^{mn} R_{\nu\sigma mn} R_{\alpha\kappa}^{ab} R_{\beta\lambda}^{cd} - \frac{1}{2} R_{\alpha\kappa}^{ab} R_{\beta\lambda}^{cd} R_{\rho\sigma}^{mn} R_{\mu\nu mn} \right] + X .\end{aligned}\quad (4.50)$$

In  $X$  we collect all terms that are not written in an explicitly covariant way

$$\begin{aligned}X &= \frac{l^2}{1024\pi G_N} \theta^{\kappa\lambda} \theta^{\rho\sigma} \epsilon^{\mu\nu\alpha\beta} \epsilon_{mnpq} \int d^4x \left[ -R_{\alpha\beta}^{ab} R_{\mu\kappa ab} \omega_\rho^{mn} (D_\sigma R_{\nu\lambda})^{pq} \right. \\ &\quad \left. - \frac{1}{2} R_{\mu\rho}^{ab} R_{\nu\sigma ab} \omega_\kappa^{mn} (D_\lambda R_{\alpha\beta})^{pq} - \frac{1}{4} R_{\rho\sigma e}^b R_{\mu\nu}^{eb} \omega_\kappa^{mn} (D_\lambda R_{\alpha\beta})^{pq} \right] .\end{aligned}\quad (4.51)$$

This term can be rewritten in the following form

$$\begin{aligned}X &= \frac{l^2}{1024\pi G_N} \cdot \frac{1}{2} \theta^{\kappa\lambda} \epsilon^{\mu\nu\alpha\beta} \epsilon_{mnpq} \omega_\kappa^{mn} (D_\lambda R_{\alpha\beta})^{pq} R_{\mu\rho}^{ab} R_{\nu\sigma ab} \\ &\quad \left( \theta^{\rho\nu} \epsilon^{\mu\sigma\alpha\beta} + \theta^{\rho\mu} \epsilon^{\sigma\alpha\beta\nu} + \theta^{\rho\sigma} \epsilon^{\alpha\beta\nu\mu} + \theta^{\rho\alpha} \epsilon^{\beta\nu\mu\sigma} + \theta^{\rho\beta} \epsilon^{\nu\mu\sigma\alpha} \right) .\end{aligned}$$

The expression in the bracket is a totally antisymmetric quantity with five indices in four dimensional Minkowski space and therefore it vanishes. Finally, the second order correction to the Gauss-Bonnet action, written in a gauge covariant way is

$$\begin{aligned}S_{GB}^{(2)} &= -\frac{l^2}{1024\pi G_N} \theta^{\kappa\lambda} \theta^{\rho\sigma} \epsilon^{\mu\nu\alpha\beta} \epsilon_{abcd} \int d^4x \left[ R_{\alpha\beta}^{cd} R_{\mu\kappa}^{ab} R_{\nu\rho}^{mn} R_{\lambda\sigma mn} \right. \\ &\quad - \frac{1}{2} R_{\alpha\beta}^{cd} R_{\mu\kappa}^{ab} R_{\nu\lambda}^{mn} R_{\rho\sigma mn} + R_{\alpha\beta}^{mn} R_{\mu\kappa mn} R_{\nu\rho}^{ab} R_{\lambda\sigma}^{cd} \\ &\quad \left. + R_{\mu\rho}^{mn} R_{\nu\sigma mn} R_{\alpha\kappa}^{ab} R_{\beta\lambda}^{cd} - \frac{1}{2} R_{\alpha\kappa}^{ab} R_{\beta\lambda}^{cd} R_{\rho\sigma}^{mn} R_{\mu\nu mn} \right] .\end{aligned}\quad (4.52)$$

We see that the correction is of fourth order in the curvature. Although it is a correction of a topological term, the correction itself seems not to be topological and it would be interesting to see how it modifies the equations of motion.

#### 4.2.2 Cosmological constant action

The second order correction to the cosmological constant action is given by

$$S_\Lambda^{(2)} = -\frac{i}{256\pi G_N l^2} \epsilon^{\mu\nu\rho\sigma} \int d^4x \text{Tr} \left( \hat{E}_\mu \star \hat{E}_\nu \star \hat{E}_\rho \star \hat{E}_\sigma \gamma_5 \right)^{(2)}$$

$$\begin{aligned}
&= -\frac{i}{256\pi G_N l^2} \epsilon^{\mu\nu\rho\sigma} \int d^4x \text{Tr} \left( 4e_\mu e_\nu e_\rho \hat{E}_\sigma^{(2)} + 4e_\mu e_\nu \hat{E}_\rho^{(1)} \hat{E}_\sigma^{(1)} \right. \\
&\quad + 2e_\mu \hat{E}_\nu^{(1)} e_\rho \hat{E}_\sigma^{(1)} + i\theta^{\alpha\beta} \{ \partial_\alpha \hat{E}_\mu^{(1)}, \partial_\beta e_\nu \} e_\rho e_\sigma \\
&\quad - \frac{1}{4} \theta^{\alpha\beta} \theta^{\kappa\lambda} ((\partial_\alpha \partial_\kappa e_\mu)(\partial_\beta \partial_\lambda e_\nu) e_\rho e_\sigma + (\partial_\alpha e_\mu)(\partial_\beta e_\nu)(\partial_\kappa e_\rho)(\partial_\lambda e_\sigma)) \\
&\quad \left. + \frac{i}{2} \theta^{\alpha\beta} \{ \partial_\alpha e_\mu (\partial_\beta e_\nu), [\hat{E}_\rho^{(1)}, e_\sigma] \} \right) \gamma_5 . \tag{4.53}
\end{aligned}$$

Now we insert the solutions for  $\hat{E}_\mu^{(1)}$  and  $\hat{E}_\mu^{(2)}$  given in Appendix B and calculate the traces. As in the previous case, we simplify the calculation using the trick of constant fields. This time however, we take only the commutative vielbein  $e_\mu$  to be constant while the commutative gauge field  $\omega_\mu$  is arbitrary. This leads to the action

$$\begin{aligned}
S_\Lambda^{(2)} &= \frac{1}{16\pi G_N l^2} \theta^{\kappa\lambda} \theta^{\alpha\beta} \int d^4x e \left( 6e_a^\mu \hat{E}_\mu^{(2)a} - 32(e_a^\mu e_b^\nu - e_a^\nu e_b^\mu) \hat{E}_\mu^{(1)a} \hat{E}_\nu^{(1)b} \right) \\
&= -\frac{1}{256\pi G_N l^2} \theta^{\kappa\lambda} \theta^{\alpha\beta} \int d^4x e \left( 3\partial_\lambda \omega_\kappa^{cd} \omega_\alpha^{ac} \omega_\beta^{da} + 3\partial_\lambda \omega_\kappa^{cd} \omega_\alpha^{da} \omega_\beta^{ca} \right. \\
&\quad \left. + 2\omega_\kappa^{ab} \omega_\alpha^{ba} \omega_\lambda^{cd} \omega_\beta^{dc} + 8\omega_\kappa^{ab} \omega_\lambda^{bc} \omega_\alpha^{cd} \omega_\beta^{da} \right) , \tag{4.54}
\end{aligned}$$

where in the last line we contracted the  $\epsilon$ -symbols. Also, the commutative vielbein  $e_\mu$  is used to convert  $SO(1,3)$  indices into coordinate indices  $e_\mu^a X_a = X_\mu$ . The local index  $a, b, \dots$  are raised by the metric  $\eta^{ab}$ . The inverse vielbein  $e_a^\mu$  is defined by  $e_a^\mu e_b^\mu = \delta_b^a$ .

This expression can be rewritten in a covariant form

$$\begin{aligned}
S_\Lambda^{(2)} &= -\frac{1}{512\pi G_N l^2} \theta^{\kappa\lambda} \theta^{\alpha\beta} \int d^4x e \left( 6R_{\kappa\alpha}^{ab} R_{\lambda\beta ab} - 3R_{\alpha\beta}^{ab} R_{\kappa\lambda ab} \right. \\
&\quad + 4R_{\alpha\beta}^{\gamma\delta} (D_\kappa e_\mu)^a (D_\lambda e_\gamma)^b (e_a^\mu e_{\delta b} + e_b^\mu e_{\delta a}) \\
&\quad - 4R_{\alpha\beta}^{\gamma\delta} (D_\kappa e_\gamma)^a (D_\lambda e_\delta)_a + 4(D_\kappa D_\alpha e_\mu)^a (D_\lambda D_\beta e_\nu)^b (e_a^\mu e_b^\nu - e_b^\mu e_a^\nu) \\
&\quad \left. - 8R_{\kappa\alpha}^{\gamma\delta} (D_\beta e_\gamma)^a (D_\lambda e_\delta)_a + 8R_{\kappa\alpha}^{\gamma\delta} (D_\beta e_\gamma)^a (D_\lambda e_\mu)^b (e_a^\mu e_{\delta b} + e_b^\mu e_{\delta a}) \right) . \tag{4.55}
\end{aligned}$$

The correction has terms that are second, first and zeroth order in the curvature.

### 4.2.3 Einstein-Hilbert action

Finally, we look at the second order correction for the Einstein-Hilbert action. It is given by

$$S_{EH}^{(2)} = \frac{1}{64\pi G_N} \epsilon^{\mu\nu\rho\sigma} \int d^4x \text{Tr} \left( \hat{E}_\mu \star \hat{E}_\nu \star \hat{R}_{\rho\sigma} \gamma_5 \right)^{(2)} . \tag{4.56}$$

Expanding the noncommutative fields and the  $\star$ -product up to second order in NC parameter  $\theta^{\mu\nu}$  we obtain

$$S_{EH}^{(2)} = \frac{1}{64\pi G_N} \epsilon^{\mu\nu\rho\sigma} \int d^4x \text{Tr} \left( \frac{1}{2} [e_\mu, e_\nu] \hat{R}_{\rho\sigma}^{(2)} \gamma_5 + [\hat{E}_\mu^{(1)}, e_\nu] \hat{R}_{\rho\sigma}^{(1)} \gamma_5 \right)$$

$$\begin{aligned}
& + \frac{i}{2} \theta^{\alpha\beta} \{ \partial_\alpha e_\mu, \partial_\beta e_\nu \} \hat{R}_{\rho\sigma}^{(1)} \gamma_5 + [\hat{E}_\mu^{(2)}, e_\nu] R_{\rho\sigma} \gamma_5 + \frac{1}{2} [\hat{E}_\mu^{(1)}, \hat{E}_\nu^{(1)}] R_{\rho\sigma} \gamma_5 \\
& + \frac{i}{2} \theta^{\alpha\beta} \{ \partial_\alpha e_\mu, \partial_\beta \hat{E}_\nu^{(1)} \} R_{\rho\sigma} \gamma_5 - \frac{1}{16} \theta^{\alpha\beta} \theta^{\gamma\delta} [\partial_\alpha \partial_\gamma e_\mu, \partial_\beta \partial_\delta e_\nu] R_{\rho\sigma} \gamma_5 \Big). \quad (4.57)
\end{aligned}$$

The next step is to insert the explicit SW map solutions for the fields. This leads to a very complicated expression which is not written in a gauge covariant way. The SW map guarantees that the final result has to be covariant under the commutative  $SO(1,3)$  gauge symmetry. We try to apply the same trick as before when calculating the corrections to the Gauss-Bonnet and the cosmological constant actions. Unfortunately, we notice that we cannot reconstruct all the covariant terms uniquely. For example, a term of the form

$$\epsilon^{\mu\nu\rho\sigma} \theta^{\alpha\beta} \theta^{\kappa\lambda} \int d^4x \text{Tr}(\omega_\alpha \omega_\beta \omega_\kappa \omega_\lambda \omega_\mu \omega_\nu e_\rho e_\sigma \gamma_5) \quad (4.58)$$

can be recognized as

$$\epsilon^{\mu\nu\rho\sigma} \theta^{\alpha\beta} \theta^{\kappa\lambda} \int d^4x \text{Tr}(R_{\alpha\beta} R_{\kappa\lambda} R_{\mu\nu} e_\rho e_\sigma \gamma_5), \quad (4.59)$$

but it can also be seen as one part of

$$\epsilon^{\mu\nu\rho\sigma} \theta^{\alpha\beta} \theta^{\kappa\lambda} \int d^4x \text{Tr}((D_\kappa R_{\alpha\beta})(D_\lambda R_{\mu\nu}) e_\rho e_\sigma \gamma_5), \quad (4.60)$$

Therefore, to avoid this ambiguity we use the method developed in [24] to calculate the second order correction for the action (4.56). The main idea of this method is to write the SW map solutions for composite fields and to simplify calculations in that way.

For example, the  $\star$ -product of two noncommutative vielbeins is

$$\hat{E}_\mu \star \hat{E}_\nu = e_\mu e_\nu + (\hat{E}_\mu \star \hat{E}_\nu)^{(1)} + (\hat{E}_\mu \star \hat{E}_\nu)^{(2)} + \dots \quad (4.61)$$

Using (4.42) one can check that this product transforms in adjoint representation of the gauge group. The first order of (4.61)

$$(\hat{E}_\mu \star \hat{E}_\nu)^{(1)} = \hat{E}_\mu^{(1)} e_\nu + e_\mu \hat{E}_\nu^{(1)} + \frac{i}{2} \theta^{\alpha\beta} \partial_\alpha e_\mu \partial_\beta e_\nu \quad (4.62)$$

can be rewritten in the following form

$$(\hat{E}_\mu \star \hat{E}_\nu)^{(1)} = -\frac{1}{4} \theta^{\alpha\beta} \{ \omega_\alpha, \partial_\beta (e_\mu e_\nu) + D_\beta (e_\mu e_\nu) \} + \frac{i}{2} \theta^{\alpha\beta} (D_\alpha e_\mu)(D_\beta e_\nu). \quad (4.63)$$

The calculation is straightforward. Notice that the first term in (4.63) is a solution of the SW map for the field  $\hat{\psi} = \hat{E}_\mu \star \hat{E}_\nu$  in the adjoint representation, compare with (4.43). The second term appears because the field  $\hat{\psi} = \hat{E}_\mu \star \hat{E}_\nu$  is not a fundamental field but a product of two fundamental fields. Also notice that the second term is

written in terms of covariant derivatives. This will be a big advantage when we write the action (4.56) in a gauge covariant form.

One can generalize (4.63) and write an expression that is valid to all orders. We will not do that here, for details look at [24].

In the same way we can find the first order term of  $\hat{R}_{\mu\nu} \star \hat{E}_\rho \star \hat{E}_\sigma$ . We consider  $\hat{R}_{\mu\nu} \star \hat{E}_\rho \star \hat{E}_\sigma$  as a  $\star$ -product of the curvature tensor  $\hat{R}_{\mu\nu}$  and the composite field  $\hat{E}_\rho \star \hat{E}_\sigma$ . Then

$$\begin{aligned} \hat{R}_{\mu\nu} \star \hat{E}_\rho \star \hat{E}_\sigma &= \hat{R}_{\mu\nu}^{(1)}(e_\rho e_\sigma) + R_{\mu\nu}(\hat{E}_\rho \star \hat{E}_\sigma)^{(1)} + \frac{i}{2}\theta^{\alpha\beta}\partial_\alpha(R_{\mu\nu})\partial_\beta(e_\rho e_\sigma) \\ &= -\frac{1}{4}\theta^{\alpha\beta}\{\omega_\alpha, \partial_\beta(R_{\mu\nu}e_\rho e_\sigma) + D_\beta(R_{\mu\nu}e_\rho e_\sigma)\} \\ &\quad + \frac{i}{2}\theta^{\alpha\beta}(D_\alpha R_{\mu\nu})D_\beta(e_\rho e_\sigma) \\ &\quad + \frac{1}{2}\theta^{\alpha\beta}\{R_{\alpha\mu}, R_{\beta\nu}\}e_\rho e_\sigma + \frac{i}{2}\theta^{\alpha\beta}R_{\mu\nu}(D_\alpha e_\rho)(D_\beta e_\sigma). \end{aligned} \quad (4.64)$$

The first order correction of Einstein-Hilbert action is

$$S_{EH}^{(1)} = \frac{1}{64\pi G_N}\epsilon^{\mu\nu\rho\sigma}\int d^4x \text{Tr}\left(\hat{R}_{\mu\nu} \star (\hat{E}_\rho \star \hat{E}_\sigma)\gamma_5\right)^{(1)}. \quad (4.65)$$

Inserting (4.64) in (4.65) and integrating by parts we obtain

$$\begin{aligned} S_{EH}^{(1)} &= -\frac{1}{256\pi G_N}\epsilon^{\mu\nu\rho\sigma}\theta^{\alpha\beta}\int d^4x \text{Tr}\gamma_5\left(\{R_{\alpha\beta}, R_{\mu\nu}\}e_\rho e_\sigma \right. \\ &\quad \left. - 2\{R_{\alpha\mu}, R_{\beta\nu}\}e_\rho e_\sigma - 2iR_{\mu\nu}(D_\alpha e_\rho)(D_\beta e_\sigma)\right). \end{aligned} \quad (4.66)$$

This expression is the same as the second line in (4.46), but (4.66) is written in a manifestly gauge covariant way. As we have said before, after taking the traces the correction (4.66) vanishes.

The second order correction of the Einstein-Hilbert action is generated from the first order correction (4.66) as

$$\begin{aligned} S_{EH}^{(2)} &= -\frac{1}{512\pi G_N}\epsilon^{\mu\nu\rho\sigma}\theta^{\alpha\beta}\int d^4x \text{Tr}\gamma_5\left(\{\hat{R}_{\alpha\beta} \star \hat{R}_{\mu\nu}\} \star \hat{E}_\rho \star \hat{E}_\sigma \right. \\ &\quad \left. - 2\{\hat{R}_{\alpha\mu} \star \hat{R}_{\beta\nu}\} \star \hat{E}_\rho \star \hat{E}_\sigma - 2i\hat{R}_{\mu\nu} \star (D_\alpha \hat{E}_\rho) \star (D_\beta \hat{E}_\sigma)\right)^{(1)}. \end{aligned} \quad (4.67)$$

Applying

$$\begin{aligned} (\hat{R}_{\alpha\beta} \star \hat{R}_{\mu\nu})^{(1)} &= -\frac{1}{4}\theta^{\kappa\lambda}\{\omega_\kappa, \partial_\lambda(R_{\alpha\beta}R_{\mu\nu}) + D_\lambda(R_{\alpha\beta}R_{\mu\nu})\} \\ &\quad + \frac{i}{2}\theta^{\kappa\lambda}(D_\kappa R_{\alpha\beta})(D_\lambda R_{\mu\nu}) + \frac{1}{2}\theta^{\kappa\lambda}(\{R_{\kappa\alpha}, R_{\lambda\beta}\}R_{\mu\nu} \\ &\quad + R_{\alpha\beta}\{R_{\kappa\mu}, R_{\lambda\nu}\}) \end{aligned}$$

and

$$\begin{aligned}
(D_\alpha \hat{E}_\rho)^{(1)} &= -\frac{1}{4}\theta^{\kappa\lambda}\{\omega_\kappa, \partial_\lambda(D_\alpha e_\rho) + D_\lambda(D_\alpha e_\rho)\} + \frac{1}{2}\theta^{\kappa\lambda}\{R_{\kappa\alpha}, D_\lambda e_\rho\} \\
(D_\alpha \hat{E}_\rho \star D_\beta \hat{E}_\sigma)^{(1)} &= -\frac{1}{4}\theta^{\kappa\lambda}\{\omega_\kappa, \partial_\lambda(D_\alpha e_\rho D_\beta e_\sigma) + D_\lambda(D_\alpha e_\rho D_\beta e_\sigma)\} \\
&\quad + \frac{i}{2}\theta^{\kappa\lambda}(D_\kappa D_\alpha e_\rho)(D_\lambda D_\beta e_\sigma) \\
&\quad + \frac{1}{2}\theta^{\kappa\lambda}\left(\{R_{\kappa\alpha}, D_\lambda e_\rho\}(D_\beta e_\sigma) + (D_\alpha e_\sigma)\{R_{\kappa\beta}, D_\lambda e_\sigma\}\right)
\end{aligned}$$

we obtain

$$\begin{aligned}
S_{EH}^{(2)} &= -\frac{1}{512\pi G_N}\epsilon^{\mu\nu\rho\sigma}\theta^{\alpha\beta}\theta^{\kappa\lambda}\int d^4x \text{Tr}\gamma_5\left(\left(-\frac{1}{4}\{R_{\kappa\lambda}, \{R_{\alpha\beta}, R_{\mu\nu}\}\}\right.\right. \\
&\quad \left.\left.+ \{R_{\kappa\lambda}, \{R_{\alpha\mu}, R_{\beta\nu}\}\} + \frac{1}{2}\{R_{\mu\nu}, \{R_{\kappa\alpha}, R_{\lambda\beta}\}\} - 2\{R_{\alpha\mu}, \{R_{\kappa\beta}, R_{\lambda\nu}\}\}\right. \\
&\quad \left.+ \frac{i}{2}[D_\kappa R_{\alpha\beta}, D_\lambda R_{\mu\nu}] - i[D_\kappa R_{\alpha\mu}, D_\lambda R_{\beta\nu}]\right)e_\rho e_\sigma \\
&\quad \left.+ i(\{R_{\alpha\beta}, R_{\mu\nu}\} - 2\{R_{\alpha\mu}, R_{\beta\nu}\})(D_\kappa e_\rho)(D_\lambda e_\sigma) - iR_{\mu\nu}\{D_\alpha e_\rho, \{R_{\kappa\beta}, D_\lambda e_\sigma\}\}\right. \\
&\quad \left.+ R_{\mu\nu}(D_\kappa D_\alpha e_\rho)(D_\lambda D_\beta e_\sigma)\right). \tag{4.68}
\end{aligned}$$

The next step is to calculate traces. Using the identities for  $\gamma$ -matrices given in Appendix A we obtain

$$\begin{aligned}
S_{EH}^{(2)} &= -\frac{1}{512\pi G_N}\theta^{\alpha\beta}\theta^{\kappa\lambda}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\int d^4x\left(e_\rho^c e_\sigma^d\left(-\frac{1}{8}R_{\kappa\lambda}{}^{ab}R_{\alpha\beta}{}^{mn}R_{\mu\nu mn}\right.\right. \\
&\quad \left.+\frac{1}{2}R_{\kappa\lambda}{}^{ab}R_{\alpha\mu}{}^{mn}R_{\beta\nu mn}+\frac{1}{4}R_{\mu\nu}{}^{ab}R_{\kappa\alpha}{}^{mn}R_{\lambda\beta mn}-R_{\alpha\mu}{}^{ab}R_{\kappa\beta}{}^{mn}R_{\lambda\nu mn}\right) \\
&\quad -\frac{1}{8}R_{\kappa\lambda\rho\sigma}R_{\alpha\beta}{}^{ab}R_{\mu\nu}{}^{cd}+\frac{1}{2}R_{\kappa\lambda\rho\sigma}R_{\alpha\mu}{}^{ab}R_{\beta\nu}{}^{cd} \\
&\quad +\frac{1}{4}R_{\mu\nu\rho\sigma}R_{\kappa\alpha}{}^{ab}R_{\lambda\beta}{}^{cd}-R_{\alpha\mu\rho\sigma}R_{\kappa\beta}{}^{ab}R_{\lambda\nu}{}^{cd} \\
&\quad +e_\rho^c e_\sigma^d(-(D_\kappa R_{\alpha\beta})^{ma}(D_\lambda R_{\mu\nu})^b{}_m+2(D_\kappa R_{\alpha\mu})^{ma}(D_\lambda R_{\beta\nu})^b{}_m) \\
&\quad + (R_{\alpha\mu}{}^{ab}R_{\beta\nu}{}^{cd}-\frac{1}{2}R_{\alpha\beta}{}^{ab}R_{\mu\nu}{}^{cd})(D_\kappa e_\rho)^m(D_\lambda e_\sigma)_m \\
&\quad \left.+2R_{\mu\nu}{}^{md}R_{\kappa\beta}{}^{ab}(D_\alpha e_\rho)_m(D_\lambda e_\sigma)^c-iR_{\mu\nu}{}^{ab}(D_\kappa D_\alpha e_\rho)^c(D_\lambda D_\beta e_\sigma)^d\right). \tag{4.69}
\end{aligned}$$

As in the case of the cosmological constant term we can contract the  $\epsilon$ -symbols and simplify (4.69). This gives

$$\begin{aligned}
S_{EH}^{(2)} &= -\frac{1}{512\pi G_N}\theta^{\alpha\beta}\theta^{\kappa\lambda}\int d^4x e\left(\frac{1}{2}R_{\kappa\lambda}{}^{\mu\nu}R_{\alpha\beta}{}^{\gamma\delta}R_{\mu\nu\gamma\delta}\right. \\
&\quad \left.+ R_{\kappa\lambda\rho\sigma}\left(\frac{1}{2}R_{\alpha\beta}{}^{\mu\nu}R_{\mu\nu}{}^{\rho\sigma}-2R_{\alpha\beta}{}^{\mu\sigma}R_\mu{}^\rho+\frac{1}{2}RR_{\alpha\beta}{}^{\rho\sigma}\right.\right. \\
&\quad \left.\left.+4R_{\beta\nu}{}^{\rho\sigma}R_\alpha{}^\nu+4R_\alpha{}^\rho R_\beta{}^\sigma-4R_{\alpha\mu}{}^{\nu\rho}R_{\beta\nu}{}^{\mu\sigma}\right)\right)
\end{aligned}$$



$$\begin{aligned}
& -2R_{\kappa\lambda}^{\mu\nu}R_{\alpha\mu}^{\gamma\delta}R_{\beta\nu\gamma\delta} - RR_{\kappa\alpha}^{\gamma\delta}R_{\lambda\beta\gamma\delta} \\
& -R_{\mu\nu\rho\sigma}(2R_{\kappa\alpha}^{\mu\nu}R_{\lambda\beta}^{\rho\sigma} + 4R_{\kappa\alpha}^{\mu\sigma}R_{\lambda\beta}^{\nu\rho}) \\
& -4R_{\alpha}^{\nu}R_{\kappa\beta}^{\gamma\delta}R_{\lambda\nu\gamma\delta} + 2R_{\alpha\mu\rho\sigma}(2R_{\kappa\beta}^{\mu\nu}R_{\lambda\nu}^{\rho\sigma} \\
& -4R_{\lambda}^{\rho}R_{\kappa\beta}^{\mu\sigma} + 4R_{\kappa\beta}^{\nu\rho}R_{\lambda\nu}^{\mu\sigma} + 2R_{\kappa\beta}^{\rho\sigma}R_{\lambda}^{\mu}) \\
& +4e_b^{\mu}e_c^{\nu}((D_{\kappa}R_{\alpha\beta})^{mb}(D_{\lambda}R_{\mu\nu})^c_m - 2(D_{\kappa}R_{\alpha\mu})^{mb}(D_{\lambda}R_{\beta\nu})^c_m) \\
& +(D_{\kappa}e_{\rho})^m(D_{\lambda}e_{\sigma})_m(2R_{\alpha\beta}^{\mu\nu}R_{\mu\nu}^{\rho\sigma} + 4R_{\alpha\beta}^{\mu\rho}R_{\mu}^{\sigma} + 4R_{\alpha\beta}^{\nu\rho}R_{\nu}^{\sigma} + 2RR_{\alpha\beta}^{\rho\sigma} \\
& -8R_{\alpha\mu}^{\rho\sigma}R_{\beta}^{\mu} - 8R_{\beta}^{\rho}R_{\alpha}^{\sigma} - 4R_{\alpha\mu}^{\nu\rho}R_{\beta\nu}^{\mu\sigma} - 4R_{\alpha\mu}^{\sigma\nu}R_{\beta\nu}^{\mu\rho}) \\
& +2i(D_{\kappa}D_{\alpha}e_{\rho})^c(D_{\lambda}D_{\beta}e_{\sigma})^d(2Re_c^{\rho}e_d^{\sigma} - 3R_{\nu}^{\rho}e_c^{\nu}e_d^{\sigma} + 3R_{\nu}^{\sigma}e_c^{\nu}e_d^{\rho} \\
& +R_{\nu}^{\rho}e_c^{\sigma}e_d^{\nu} - R_{\nu}^{\sigma}e_c^{\rho}e_d^{\nu} + 2R_{\mu\nu}^{\rho\sigma}e_c^{\mu}e_d^{\nu}) \\
& -\frac{1}{512\pi G_N}\theta^{\alpha\beta}\theta^{\kappa\lambda}\epsilon^{\mu\nu\rho\sigma}\epsilon_{abcd}\int d^4x R_{\mu\nu}^{\quad rd}R_{\kappa\beta}^{\quad ab}(D_{\alpha}e_{\rho})_r(D_{\lambda}e_{\sigma})^c. \quad (4.70)
\end{aligned}$$

The second order correction to the Einstein-Hilbert action is of the 3rd, 2nd and 1st order in the curvature. Its implications to the equations of motion have to be investigated carefully. Since the higher powers of curvature enter (4.70), it is obvious that the spin connection propagates and it cannot be expressed in terms of vielbeins.

## 5 Conclusion

Starting from the  $SO(2,3)$  gravity theory on four dimensional Minkowski space, we constructed a NC gravity theory. The starting point was the action (2.19) obtained after the spontaneous symmetry breaking of  $SO(2,3)$  to  $SO(1,3)$ . The construction is done as an expansion in the NC parameter, using the SW map. As expected, the first order correction vanishes and for the second order correction one obtains a very complicated expression which not written in an explicitly gauge covariant way. Using the trick of constant fields (connection and vielbein) and the SW map for composite fields we managed to obtain manifestly gauge invariant action.

The NC action (4.52), (4.70) and (4.55) which we obtained is invariant under the commutative  $SO(1,3)$  gauge symmetry. However, the question of diffeomorphism invariance remains to be understood better. Since  $\theta^{\mu\nu}$  is constant matrix, the symmetry under the usual (commutative, undeformed) diffeomorphisms is broken. Whether the action is invariant under a deformed diffeomorphism symmetry and what that symmetry exactly is, remains to be investigated in future.

The action (4.52), (4.70) and (4.55) can be used to calculate corrections to classical solutions such as black holes and cosmological solutions. For example, it would be interesting to see how the deformation modifies the Hawking temperature of black hole radiation. In addition, one can investigate quantum behavior of the deformed theory.

One more interesting research direction is the analysis of the full  $SO(2,3)_{\star}$  NC action (3.26). Using the SW map for the composite fields one can write the second order correction to this action. The expanded action has the commutative  $SO(2,3)$

gauge symmetry. This symmetry can be spontaneously broken to the commutative  $SO(1, 3)$  symmetry. The zeroth order of the action after SSB has to be given by (2.19) but it would be interesting to compare the higher order corrections with the results obtained here.

Our NC action contains higher powers of the curvature tensor and its covariant derivatives. These terms become increasingly important as energies become higher. The prime candidates for the study of these terms are therefore effects in the early universe, where the epoch of inflation is especially interesting. Higher powers of the curvature tensor and its contractions have been early identified as possible sources of inflationary expansion in the early universe. The model of Starobinsky [25] is probably the best known representative of the class of models in which the inflationary expansion is modeled in terms of the higher order terms in the action. It is reasonable to expect that for some values of  $\theta$  inflationary dynamics should be realized as a result of terms present in the NC action (4.52), (4.70) and (4.55). The question of the existence and the properties of inflationary solutions in the NC action (4.52), (4.70) and (4.55) requires a dedicated analysis and it will be presented elsewhere.

If we assume that the mechanism determining the value of  $\theta$  in the very early (preinflationary) universe is local, i.e. that the value of  $\theta$  acquires values locally or has constant components in small local domains, then depending on the particular values of  $\theta$  components, in some of these domains the action terms (4.66) and (4.77) will lead to inflation and in some it will not. The domains with the inflationary dynamics will expand much more than those that do not inflate and their value of  $\theta$  will be associated to very large physical volumes after the inflation. In the parts of the universe which originate from the same preinflationary domain it is therefore reasonable to expect that after the inflation the components of the  $\theta$  will be constant. Although this argument puts additional weight on theories of noncommutative spaces with constant  $\theta$ , a precautionary remark is in order. The action (4.52), (4.70) and (4.55) was derived using the assumption of constant  $\theta$ . If it is to be valid in the preinflationary epoch, the domains in which  $\theta$  is constant before the inflation should be sufficiently large so that IR corrections to (4.52), (4.70) and (4.55), associated to the size of the domain, are negligible. A more precise formulation of the claim on constancy of  $\theta$  would be that the domains in postinflationary universe with constant  $\theta$  are much bigger than the domains of constancy of  $\theta$  in the preinflationary universe.

Concerning the experimental constraints on the NC parameter one can say the following: The existing studies of inflation in modified theories of gravity (such as  $f(R)$  theories and other higher-derivative theories) may not be immediately applied to constrain  $\theta$  from the action (4.52), (4.70) and (4.55). In these expressions the  $\theta$  components are contracted with the curvature tensor and its covariant derivatives and  $f(R)$  and similar theories do not include such terms. This property is a distinguishing novelty which should be useful in the experimental verification/falsification of the theory presented in this paper.

**Acknowledgement** We would like to thank Milutin Blagojević and Maja Burić for fruitful discussion and useful comments. The work of M. D. and V. R. is supported by project 171031 of the Serbian Ministry of Education and Science. The work of H. S is supported by Ministry of Education, Science and Sport of the republic of Croatia under contract No 098-0982930-2684. This paper is done in the framework of the Serbian-Croatian bilateral project "Modified gravity theories and the accelerated expansion of Universe".

## A AdS algebra and the $\gamma$ -matrices

Algebra relations<sup>3</sup>:

$$\begin{aligned}
\{M_{AB}, \Gamma_C\} &= i\epsilon_{ABCDE}M^{DE} \\
\{M_{AB}, M_{CD}\} &= \frac{i}{2}\epsilon_{ABCDE}\Gamma^E + \frac{1}{2}(\eta_{AC}\eta_{BD} - \eta_{AD}\eta_{BC}) \\
[M_{AB}, \Gamma_C] &= i(\eta_{BC}\Gamma_A - \eta_{AC}\Gamma_B) \\
\Gamma_A^\dagger &= -\gamma_0\Gamma_A\gamma_0 \\
M_{AB}^\dagger &= \gamma_0M_{AB}\gamma_0 \\
\{\sigma_{ab}, \sigma_{cd}\} &= 2(\eta_{ac}\eta_{bd} - \eta_{ad}\eta_{bc} + i\epsilon_{abcd}\gamma_5) \\
[\sigma_{ab}, \gamma_c] &= 2i(\eta_{bc}\gamma_a - \eta_{ac}\gamma_b) \\
\{\sigma_{ab}, \gamma_c\} &= 2\epsilon_{abcd}\gamma^5\gamma^d
\end{aligned} \tag{1.71}$$

Identities with traces:

$$\begin{aligned}
\text{Tr}(\Gamma_A\Gamma_B) &= 4\eta_{AB} \\
\text{Tr}(\Gamma_A) &= \text{Tr}(\Gamma_A\Gamma_B\Gamma_C) = 0 \\
\text{Tr}(\Gamma_A\Gamma_B\Gamma_C\Gamma_D) &= 4(\eta_{AB}\eta_{CD} - \eta_{AC}\eta_{BD} + \eta_{AD}\eta_{CB}) \\
\text{Tr}(\Gamma_A\Gamma_B\Gamma_C\Gamma_D\Gamma_E) &= -4i\epsilon_{ABCDE} \\
\text{Tr}(M_{AB}M_{CD}\Gamma_E) &= i\epsilon_{ABCDE} \\
\text{Tr}(M_{AB}M_{CD}) &= -\eta_{AD}\eta_{CB} + \eta_{AC}\eta_{BD}
\end{aligned} \tag{1.72}$$

## B SW map for $SO(1, 3)_\star$ gauge group

Here we list the explicit 1st and 2nd order solutions for the gauge parameter  $\hat{\Lambda}_\epsilon$ , the gauge field  $\hat{\omega}_\mu$ , the curvature tensor  $\hat{R}_{\mu\nu}$  and the vielbein  $\hat{E}_\mu$  for the  $SO(1, 3)_\star$  gauge group.

- Gauge parameter

$$\hat{\Lambda}^{(1)} = -\frac{1}{4}\theta^{\kappa\lambda}\{\omega_\kappa, \partial_\lambda\epsilon\}, \tag{2.73}$$

---

<sup>3</sup> $\epsilon^{01235} = +1, \epsilon^{0123} = 1$

$$\begin{aligned}
\hat{\Lambda}^{(2)} &= \Lambda_\epsilon^{ab(2)} \frac{\sigma_{ab}}{4} \\
&= \frac{1}{32} \theta^{\kappa\lambda} \theta^{\rho\sigma} \left( \{\omega_\rho, \{\partial_\sigma \omega_\kappa, \partial_\lambda \epsilon\}\} + \{\omega_\rho, \{\omega_\kappa, \partial_\sigma \partial_\lambda \epsilon\}\} \right. \\
&\quad \left. + \{\{\omega_\rho, \partial_\sigma \omega_\kappa\}, \partial_\lambda \epsilon\} - \{\{R_{\rho\kappa}, \omega_\sigma\}, \partial_\lambda \epsilon\} \right. \\
&\quad \left. - 2i[\partial_\rho \omega_\kappa, \partial_\sigma \partial_\lambda \epsilon] \right). \tag{2.74}
\end{aligned}$$

- Gauge field

$$\begin{aligned}
\hat{\omega}_\mu^{(1)} &= -\frac{1}{4} \theta^{\kappa\lambda} \{\omega_\kappa, \partial_\lambda \omega_\mu + R_{\lambda\mu}\} \\
&= \omega_\mu^{(1)} I + \omega_{\mu 5}^{(1)} \gamma_5, \tag{2.75}
\end{aligned}$$

with

$$\begin{aligned}
\omega_\mu^{(1)} &= -\frac{1}{16} \theta^{\kappa\lambda} \omega_\kappa^{ab} (\partial_\lambda \omega_{\mu ab} + R_{\lambda\mu ab}), \\
\omega_{\mu 5}^{(1)} &= -\frac{i}{32} \theta^{\kappa\lambda} \epsilon_{abcd} \omega_\kappa^{ab} (\partial_\lambda \omega_\mu^{cd} + R_{\lambda\mu}^{cd}). \\
\hat{\omega}_\mu^{(2)} &= \frac{1}{4} \omega_\mu^{ab(2)} \sigma_{ab} \\
&= -\frac{1}{8} \theta^{\kappa\lambda} \{\hat{\omega}_\kappa^{(1)}, \partial_\lambda \omega_\mu + R_{\lambda\mu}\} + \{\omega_\kappa, \partial_\lambda \hat{\omega}_\mu^{(1)} + \hat{R}_{\lambda\mu}^{(1)}\} \\
&\quad - \frac{i}{16} \theta^{\kappa\lambda} \theta^{\alpha\beta} [\partial_\alpha \omega_\kappa, \partial_\beta (\partial_\lambda \omega_\mu + R_{\lambda\mu})], \tag{2.76}
\end{aligned}$$

where  $\hat{R}_{\mu\nu}^{(1)}$  is the first order corrections to the curvature tensor (see below).

- Curvature tensor

$$\begin{aligned}
\hat{R}_{\mu\nu}^{(1)} &= -\frac{1}{4} \theta^{\kappa\lambda} \{\omega_\kappa, \partial_\lambda R_{\mu\nu} + D_\lambda R_{\mu\nu}\} + \frac{1}{2} \theta^{\kappa\lambda} \{R_{\mu\kappa}, R_{\nu\lambda}\} \\
&= R_{\mu\nu}^{(1)} I + R_{\mu\nu 5}^{(1)} \gamma_5, \tag{2.77}
\end{aligned}$$

where

$$\begin{aligned}
R_{\mu\nu}^{(1)} &= \frac{1}{8} \theta^{\rho\sigma} \left( R_{\mu\rho}^{ab} R_{\nu\sigma ab} - \omega_\rho^{ab} \partial_\sigma R_{\mu\nu ab} - \frac{1}{2} \omega_{\rho ab} (\omega_{\sigma e}^a R_{\mu\nu}^{eb} - \omega_{\sigma e}^b R_{\mu\nu}^{ea}) \right), \\
R_{\mu\nu 5}^{(1)} &= \frac{i}{16} \theta^{\rho\sigma} \epsilon_{abcd} \left( R_{\mu\rho}^{ab} R_{\nu\sigma}^{cd} - \omega_\rho^{ab} \partial_\sigma R_{\mu\nu}^{cd} - \omega_\rho^{ab} \omega_{\sigma e}^c R_{\mu\nu}^{ed} \right). \\
\hat{R}_{\mu\nu}^{(2)} &= \frac{1}{4} R_{\mu\nu}^{ab(2)} \sigma_{ab} \\
&= -\frac{1}{8} \theta^{\kappa\lambda} \left( \{\omega_\kappa, \partial_\lambda \hat{R}_{\mu\nu}^{(1)} + (D_\lambda \hat{R}_{\mu\nu})^1\} + \{\hat{\omega}_\kappa^{(1)}, \partial_\lambda R_{\mu\nu} + (D_\lambda R_{\mu\nu})\} \right. \\
&\quad \left. - 2\{R_{\mu\kappa}, \hat{R}_{\nu\lambda}^{(1)}\} - 2\{\hat{R}_{\mu\kappa}^{(1)}, R_{\nu\lambda}\} \right) \\
&\quad - \frac{i}{16} \theta^{\kappa\lambda} \theta^{\rho\sigma} \left( [\partial_\rho \omega_\kappa, \partial_\sigma (\partial_\lambda R_{\mu\nu} + D_\lambda R_{\mu\nu})] - 2[\partial_\rho R_{\mu\kappa}, \partial_\sigma R_{\nu\lambda}] \right), \tag{2.78}
\end{aligned}$$

where

$$\begin{aligned}
R_{\mu\nu}^{(2)ab} = & -\frac{1}{8}\theta^{\kappa\lambda}\omega_{\kappa}^{ab}(4\partial_{\lambda}R_{\mu\nu}^{(1)} + \frac{1}{4}\theta^{\alpha\beta}\partial_{\alpha}\omega_{\lambda}^{cd}\partial_{\beta}R_{\mu\nu cd}) \\
& -\frac{i}{8}\epsilon^{abpq}\omega_{\kappa}^{pq}(2\partial_{\lambda}R_{\mu\nu 5}^{(1)} + \frac{i}{16}\theta^{\alpha\beta}\epsilon_{cdef}\partial_{\alpha}\omega_{\lambda}^{cd}\partial_{\beta}R_{\mu\nu}^{ef}) \\
& -\frac{1}{4}\theta^{\kappa\lambda}\omega_{\kappa}^{(1)}(\partial_{\lambda}R_{\mu\nu}^{ab} + (D_{\lambda}R_{\mu\nu})^{ab}) \\
& -\frac{i}{8}\theta^{\kappa\lambda}\epsilon_{abcd}\omega_{\kappa 5}^{(1)}(\partial_{\lambda}R_{\mu\nu}^{cd} + (D_{\lambda}R_{\mu\nu})^{cd}) \\
& +\frac{1}{2}\theta^{\kappa\lambda}(2R_{\mu\kappa}^{ab}R_{\nu\lambda}^{(1)} + i\epsilon_{abcd}R_{\mu\kappa}^{cd}R_{\nu\lambda 5}^{(1)}) \\
& +\frac{1}{16}\theta^{\kappa\lambda}\theta^{\rho\sigma}\left(\partial_{\rho}\omega_{\kappa e}^a\partial_{\sigma}(\partial_{\lambda}R_{\mu\nu}^{eb} + (D_{\lambda}R_{\mu\nu})^{eb})\right. \\
& \left.-\partial_{\rho}\omega_{\kappa e}^b\partial_{\sigma}(\partial_{\lambda}R_{\mu\nu}^{ea} + (D_{\lambda}R_{\mu\nu})^{ea})\right. \\
& \left.-2\partial_{\rho}R_{\mu\kappa e}^a\partial_{\sigma}R_{\nu\lambda}^{eb} + 2\partial_{\rho}R_{\mu\kappa e}^b\partial_{\sigma}R_{\nu\lambda}^{ea}\right) .
\end{aligned}$$

- Vielbein

$$\begin{aligned}
\hat{E}_{\mu}^{(1)} &= -\frac{1}{4}\theta^{\kappa\lambda}\{\omega_{\kappa}, \partial_{\lambda}e_{\mu} + D_{\lambda}e_{\mu}\} \\
&= E_{\mu d 5}^{(1)}\gamma_5\gamma^d \\
&= -\frac{1}{8}\theta^{\kappa\lambda}\epsilon_{abcd}\omega_{\kappa}^{ab}(2\partial_{\lambda}e_{\mu}^c + \omega_{\lambda}^{ce}e_{\mu e})\gamma_5\gamma^d . \tag{2.79}
\end{aligned}$$

$$\begin{aligned}
\hat{E}_{\mu}^{(2)} &= E_{\mu a}^{(2)}\gamma^a \\
&= -\frac{1}{4}\theta^{\kappa\lambda}\omega_{\kappa}^{(1)}(2\partial_{\lambda}e_{\mu}^a + \omega_{\lambda}^{ab}e_{\mu b})\gamma_a \\
&\quad -\frac{1}{16}\epsilon_{cdfa}\theta^{\kappa\lambda}\omega_{\kappa}^{cd}(2\partial_{\lambda}E_{\mu 5}^{(1)f} + \omega_{\lambda}^{fe}E_{\mu e 5}^{(1)} - 2i\omega_{\lambda 5}^{(1)}e_{\mu}^f) \\
&\quad +\frac{1}{4}\theta^{\alpha\beta}\epsilon_{mne}^f\partial_{\alpha}\omega_{\lambda}^{mn}\partial_{\beta}e_{\mu}^e\gamma^a \\
&\quad +\frac{1}{16}\theta^{\kappa\lambda}\theta^{\rho\sigma}\partial_{\rho}\omega_{\kappa}^{ab}\partial_{\sigma}(\partial_{\lambda}e_{\mu}^b + \omega_{\lambda}^{bd}e_{\mu d})\gamma_a . \tag{2.80}
\end{aligned}$$

## References

- [1] A. Connes, *Non-commutative Geometry*, Academic Press (1994).  
J. Madore, *An Introduction to Noncommutative Differential Geometry and its Physical Applications*, 2nd Edition, Cambridge Univ. Press (1999).  
P. Aschieri, M. Dimitrijević, P. Kulish, F. Lizzi and J. Wess *Noncommutative spacetimes: Symmetries in noncommutative geometry and field theory*, Lecture notes in physics **774**, Springer (2009).

- [2] F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz and D. Sternheimer, *Deformation theory and quantization*, Ann. Phys. **111**, 61 (1978).  
D. Sternheimer, *Deformation quantization: Twenty years after*, AIP Conf. Proc. **453**, 107 (1998) [math.qa/9809056].  
Maxim Kontsevich, *Deformation quantization of Poisson manifolds, I*, Lett. Math. Phys. **66**, 157 (2003) [q-alg/9709040].
- [3] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, JHEP **09**, 032 (1999) [hep-th/9908142].
- [4] B. Jurčo, S. Schraml, P. Schupp and J. Wess, *Enveloping algebra valued gauge transformations for non-abelian gauge groups on non-commutative spaces*, Eur. Phys. J. **C17**, 521 (2000) [hep-th/0006246].  
B. Jurčo, L. Möller, S. Schraml, P. Schupp and J. Wess, *Construction of non-Abelian gauge theories on noncommutative spaces*, Eur. Phys. J. **C21**, 383 (2001) [hep-th/0104153].
- [5] M. Burić, D. Latas and V. Radovanović, *Renormalizability of noncommutative  $SU(N)$  gauge theory*, JHEP **0602**, 046 (2006) [hep-th/0510133].  
M. Burić, V. Radovanović and J. Trampetić, *The one-loop renormalization of the gauge sector in the noncommutative standard model*, JHEP **0703**, 030 (2007) [hep-th/0609073].
- [6] C. P. Martin, *The gauge anomaly and the Seiberg-Witten map*, Nucl. Phys. B **652**, 72 (2003) [arXiv:hep-th/0211164].
- [7] X. Calmet, B. Jurčo, P. Schupp, J. Wess and M. Wohlgenannt, *The Standard Model on noncommutative spacetime*, Eur. Phys. J. **C23**, 363 (2002) [hep-ph/0111115].  
P. Aschieri, B. Jurčo, P. Schupp and J. Wess, *Noncommutative GUTs, standard model and  $C$ ,  $P$ ,  $T$* , Nucl. Phys. B **651**, 45 (2003) [arXiv:hep-th/0205214].
- [8] W. Behr, N. G. Deshpande, G. Duplancić, P. Schupp, J. Trampetić and J. Wess, *The  $Z \rightarrow \gamma \gamma$ ,  $gg$  Decays in the Noncommutative Standard Model*, Eur. Phys. J. **C29**, 441 (2003) [hep-ph/0202121].  
B. Melić, K. Passek-Kumerički, P. Schupp, J. Trampetić and M. Wohlgenannt, *The Standard Model on Non-Commutative Space-Time: Electroweak Currents and Higgs Sector*, Eur. Phys. J. **C42**, 483 (2005) [hep-ph/0502249].  
B. Melić, K. Passek-Kumerički, P. Schupp, J. Trampetić and M. Wohlgenannt, *The Standard Model on Non-Commutative Space-time: Strong Interactions Included*, Eur. Phys. J. **C42**, 499 (2005) [hep-ph/0503064].
- [9] P. Aschieri, C. Blohmann, M. Dimitrijević, F. Meyer, P. Schupp and J. Wess, *A Gravity Theory on Noncommutative Spaces*, Class. Quant. Grav. **22**, 3511 (2005) [hep-th/0504183].

- [10] P. Aschieri, M. Dimitrijević, F. Meyer and J. Wess, *Noncommutative Geometry and Gravity*, Class. Quant. Grav. **23**, 1883 (2006) [hep-th/0510059].
- [11] M. Blagojević, *Gravitation and Gauge Symmetries*, Institute of Physics Publication, Bristol (2002).
- [12] A. H. Chamseeddine, *Deforming Einstein's gravity*, Phys. Lett. B **504** 33 (2001) [hep-th/0009153].
- [13] A. H. Chamseeddine,  *$SL(2, C)$  gravity with a complex vierbein and its noncommutative extension*, Phys. Rev. D **69**, 024015 (2004) [hep-th/0309166].
- [14] M. A. Cardella and D. Zanon, *Noncommutative deformation of four-dimensional gravity*, **20**, L95 (2003) [hep-th/0212071].
- [15] R. Banerjee, P. Mukherjee and S. Samanta, *Lie algebraic noncommutative gravity*, Phys. Rev D **75**, 125020 (2007) [hep-th/0703128].
- [16] Yang-Gang Miao, Zhao Xue and Shao-Jun Zhang,  *$U(2, 2)$  gravity on noncommutative space with symplectic structure*, Phys. Rev. D **83**, 024023 (2011) [arXiv:1006.4074]
- [17] P. Aschieri and L. Castellani, *Noncommutative  $D = 4$  gravity coupled to fermions* JHEP, **0906**, 086 (2009) [arXiv:0902.3823].
- [18] P. Aschieri and L. Castellani, *Noncommutative gauge fields coupled to noncommutative gravity*, [arXiv:1205.1911]
- [19] K. S. Stelle and P. C. West, *Spontaneously broken de Sitter symmetry and the gravitational holonomy group*, Phys. Rev D **21**, 1466 (1980).
- [20] S. W. MacDowell and F. Mansouri, *Unified geometrical theory of gravity and supergravity*, Phys. Rev. Lett. **38**, 739 (1977).
- [21] P. K. Townsend, *Small-scale structure of spacetime as the origin of the gravitation constant*, Phys. Rev. D **15**, 2795 (1977).
- [22] K. Ulker and B. Yapiskan, *Seiberg-Witten maps to all orders*, Phys. Rev. D **77**, 065006 (2008) [arXiv: 0712.0506].
- [23] P. Aschieri and L. Castellani, *Noncommutative gravity coupled to fermions: second order expansion via Seiberg-Witten map*, [arXiv:1111.4822].
- [24] P. Aschieri, L. Castellani and M. Dimitrijević, *Noncommutative gravity at second order via Seiberg-Witten map*, [arXiv:1207.4346].
- [25] A. A. Starobinsky, *A New Type of Isotropic Cosmological Models Without Singularity*, Phys. Lett. B **91**, 99 (1980).